Indirect Proofs

Outline for Today

- ***What is an Implication?***
  - Understanding a key type of mathematical statement.
- ***Negations and their Applications***
  - How do you show something is *not* true?
- ***Proof by Contrapositive***
  - What's a contrapositive?
  - And some applications!
- ***Proof by Contradiction***
  - The basic method.
  - And some applications!

Logical Implication

If n is an even integer, then n^2 is an even integer.

An ***implication*** is a statement of the form "If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

This part of the implication is called the ***antecedent***.

This part of the implication is called the ***consequent***.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

If n is an even integer, then n^2 is an even integer.

If m and n are odd integers, then $m+n$ is even.

If you like the way you look that much,
then you should go and love yourself.

An ***implication*** is a statement of the form
"If P is true, then Q is true."

What Implications Mean

“If there's a rainbow in the sky, then it's raining somewhere.”

- In mathematics, implication is directional.
  - The above statement doesn't mean that if it's raining somewhere, there has to be a rainbow.
- In mathematics, implications only say something about the consequent when the antecedent is true.
  - If there's no rainbow, it doesn't mean there's no rain.
- In mathematics, implication says nothing about causality.
  - Rainbows do not cause rain.

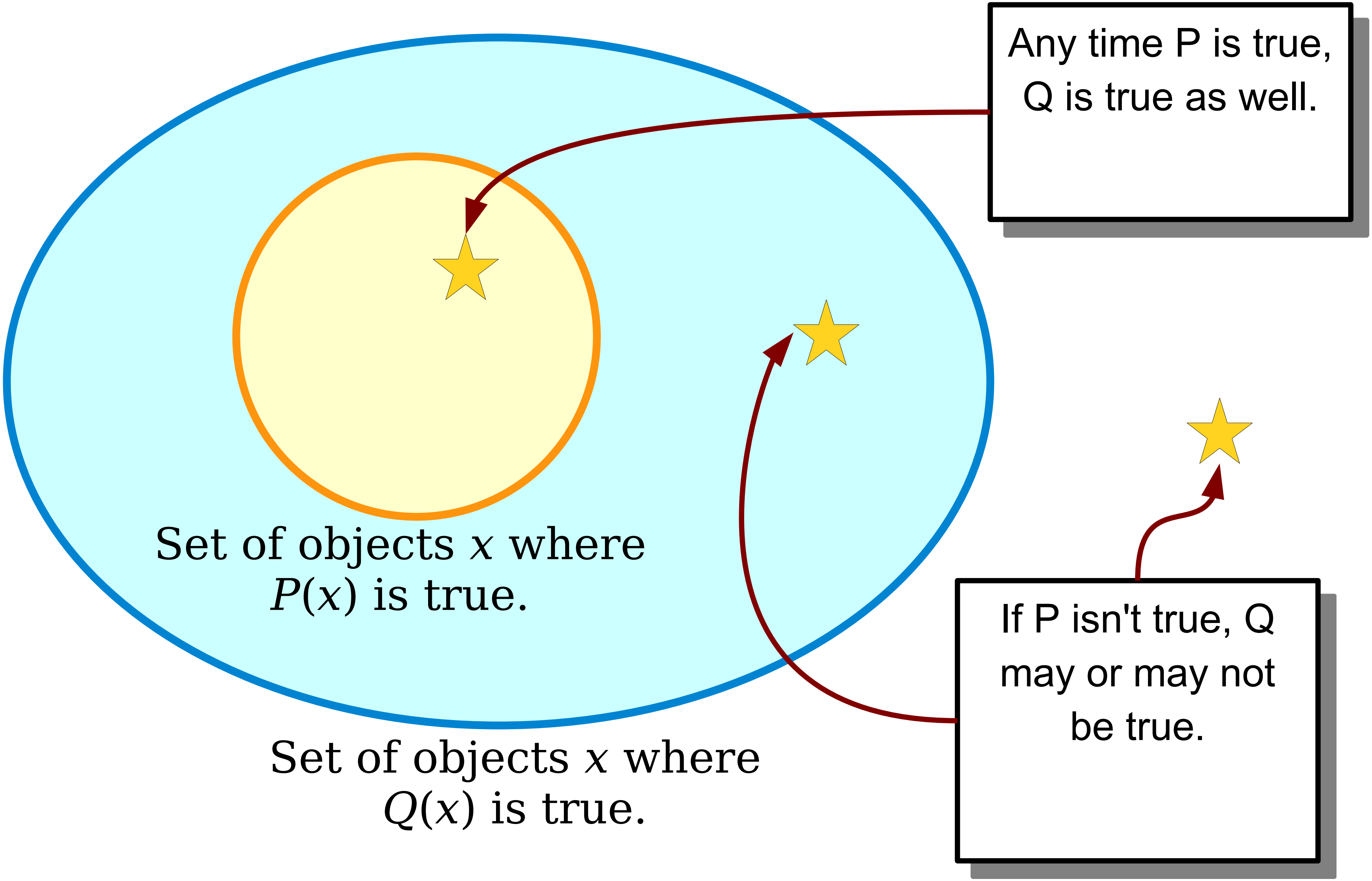
What Implications Mean

- In mathematics, a statement of the form

  For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true

  means that any time you find an object *x* where *P*(*x*) is true, you will see that *Q*(*x*) is also true (for that same *x*).
- There is no discussion of causation here. It simply means that if you find that *P*(*x*) is true, you'll find that *Q*(*x*) is also true.

Implication, Diagrammatically



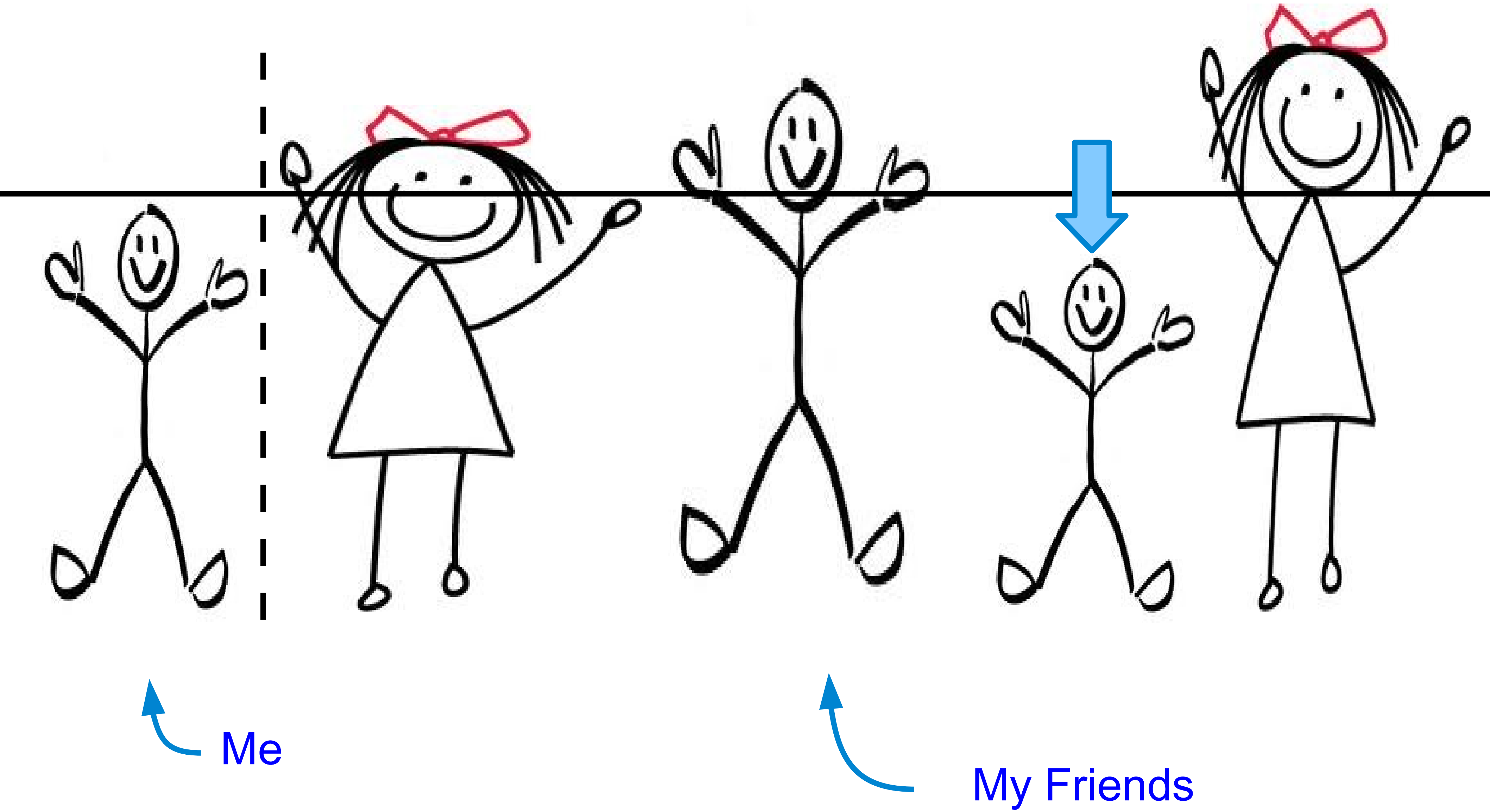
Negations

Negations

- A ***proposition*** is a statement that is either true or false.
- Some examples:
  - If n is an even integer, then n^2 is an even integer.
  - $\emptyset = \mathbb{R}$.
- The ***negation*** of a proposition X is a proposition that is true whenever X is false and is false whenever X is true.
- For example, consider the proposition “it is snowing outside.”
  - Its negation is “it is not snowing outside.”
  - Its negation is *not* “it is sunny outside.” ⚠
  - Its negation is *not* “we’re in the Bay Area.” ⚠

How do you find the negation of a statement?

“All My Friends Are Taller Than Me”



The negation of the ***universal*** statement

Every *P* is a *Q*

is the ***existential*** statement

There is a *P* that is not a *Q*.

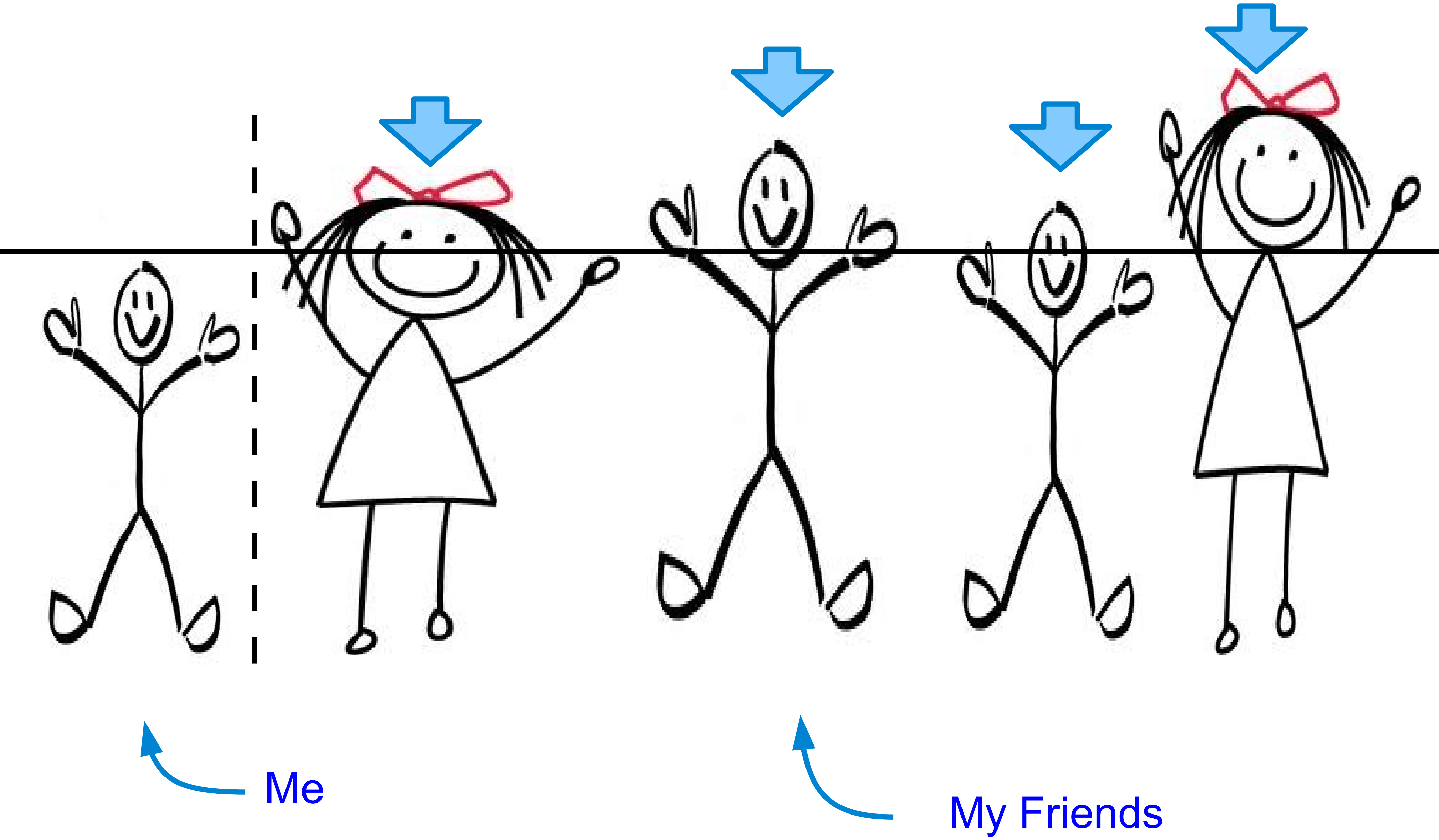
The negation of the ***universal*** statement

For all *x*, *P*(*x*) is true.

is the ***existential*** statement

There exists an *x* where *P*(*x*) is false.

“Some Friend Is Shorter Than Me”



The negation of the ***existential*** statement

There exists a *P* that is a *Q*

is the ***universal*** statement

Every *P* is not a *Q*.

The negation of the ***existential*** statement

There exists an *x* where *P*(*x*) is true

is the ***universal*** statement

For all *x*, *P*(*x*) is false.

How do you negate an implication?


Story Time!

Ancient Contract:

If Nanni pays money to Ea-Nasir, then
Ea-Nasir will give Nanni quality copper ingots.

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If Nanni pays money to Ea-Nasir, then
Ea-Nasir will give Nanni quality copper ingots.

Question: What has to happen for the contract to be broken?

A) Nanni does not pay Ea-Nasir and Ea-Nasir does not give the ingots.

B) Nanni does not pay Ea-Nasir and Ea-Nasir gives the ingots.

C) Nanni pays Ea-Nasir and Ea-Nasir does not give the ingots.

D) Nanni pays Ea-Nasir and Ea-Nasir gives the ingots.

Respond at pollev.com/zhenglian740

Ancient Contract:

If Nanni pays money to Ea-Nasir, then
Ea-Nasir will give Nanni quality copper ingots.

Question: What has to happen for this contract to be broken?
Answer: Nanni pays Ea-Nasir and doesn’t get quality copper ingots.

The negation of the statement

“For any *x*, if *P*(*x*) is true, then *Q*(*x*) is true”

is the statement

“There is at least one *x* where *P*(*x*) is true and *Q*(*x*) is false.”

The negation of an implication is not an implication!

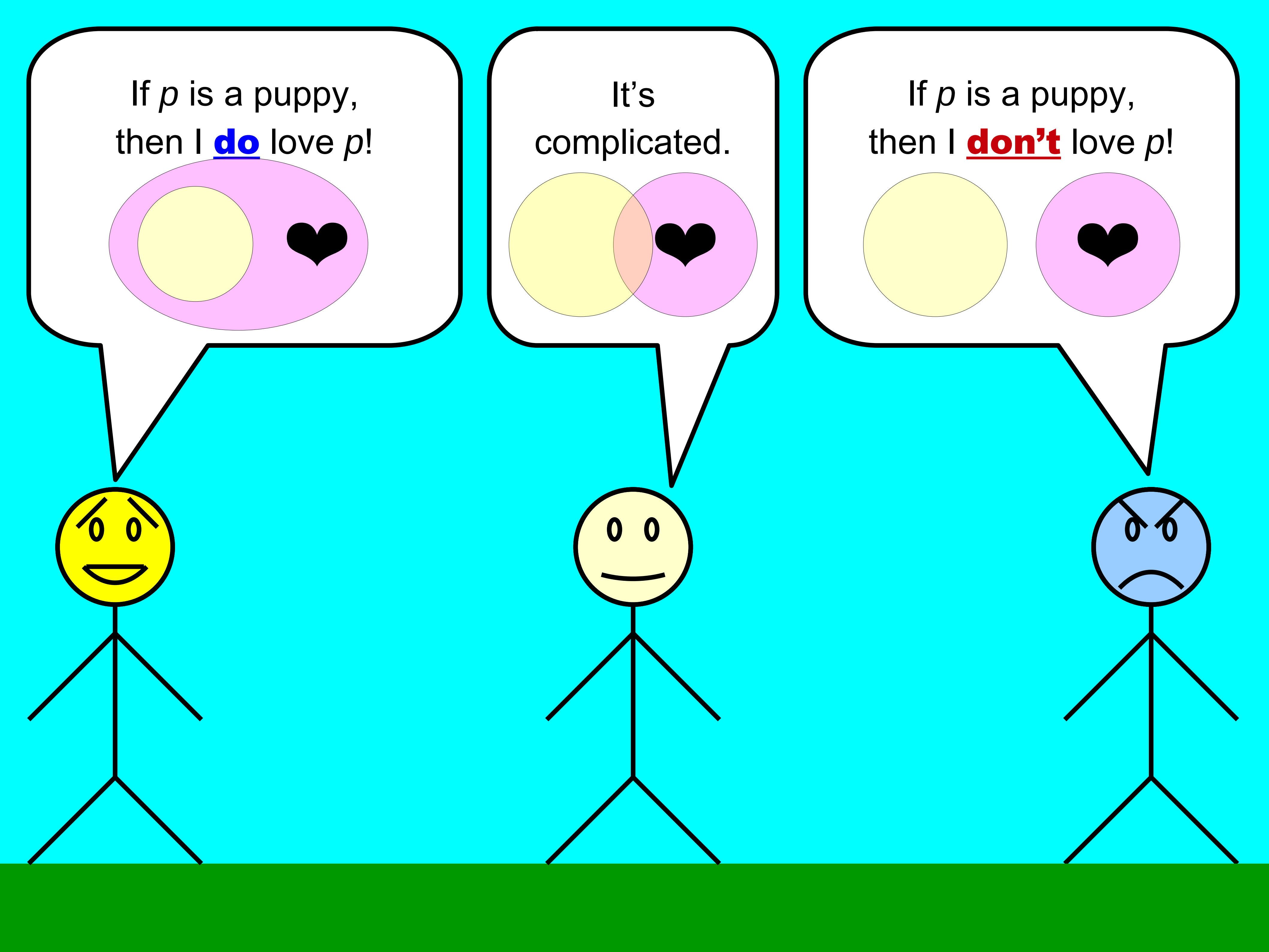
The negation of the statement

"For any x, if $P(x)$ is true, then $Q(x)$ is true"

is the statement

"There is at least one x where $P(x)$ is true and $Q(x)$ is false."

The negation of an implication is not an implication!


If *p* is a puppy, then I **do** love *p*!
It's complicated.
If *p* is a puppy, then I **don't** love *p*!

How to Negate Universal Statements:

"For all *x*, *P*(*x*) is true"

becomes

"There is an *x* where *P*(*x*) is false."

How to Negate Existential Statements:

"There exists an *x* where *P*(*x*) is true"

becomes

"For all *x*, *P*(*x*) is false."

How to Negate Implications:

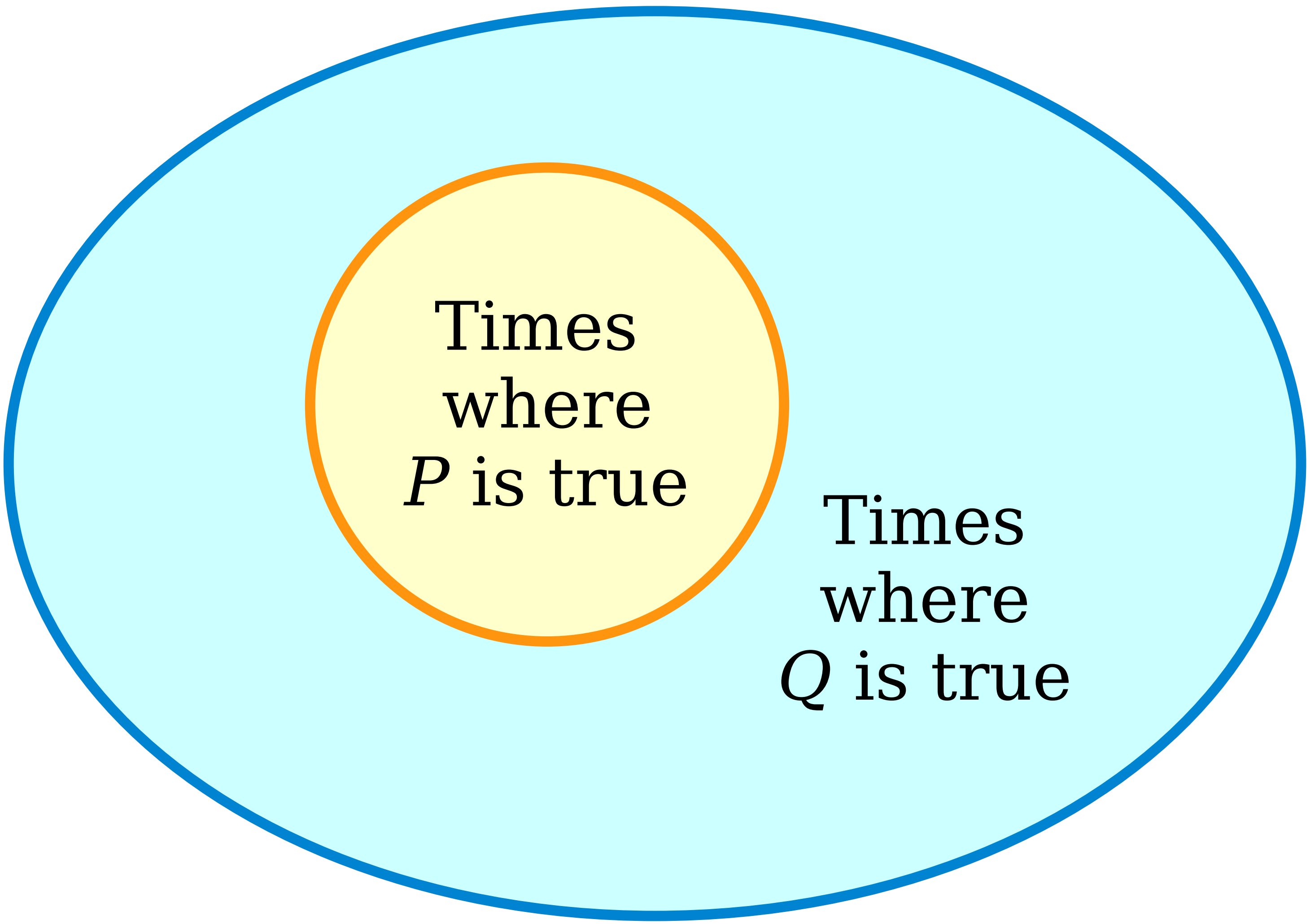
"For every *x*, if *P*(*x*) is true, then *Q*(*x*) is true"

becomes

"There is an *x* where *P*(*x*) is true and *Q*(*x*) is false."

Proof by Contrapositive

Implication, Diagrammatically



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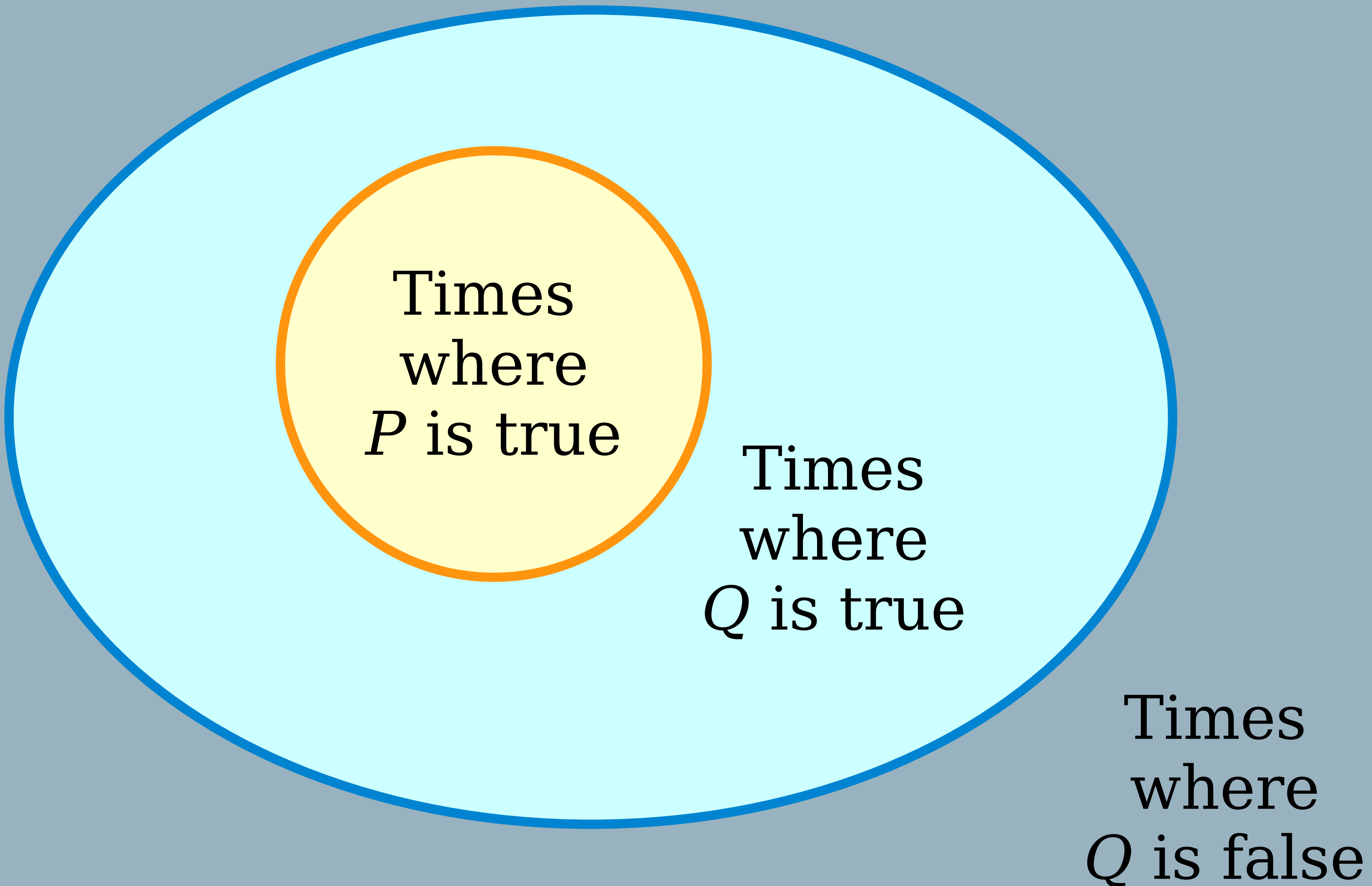
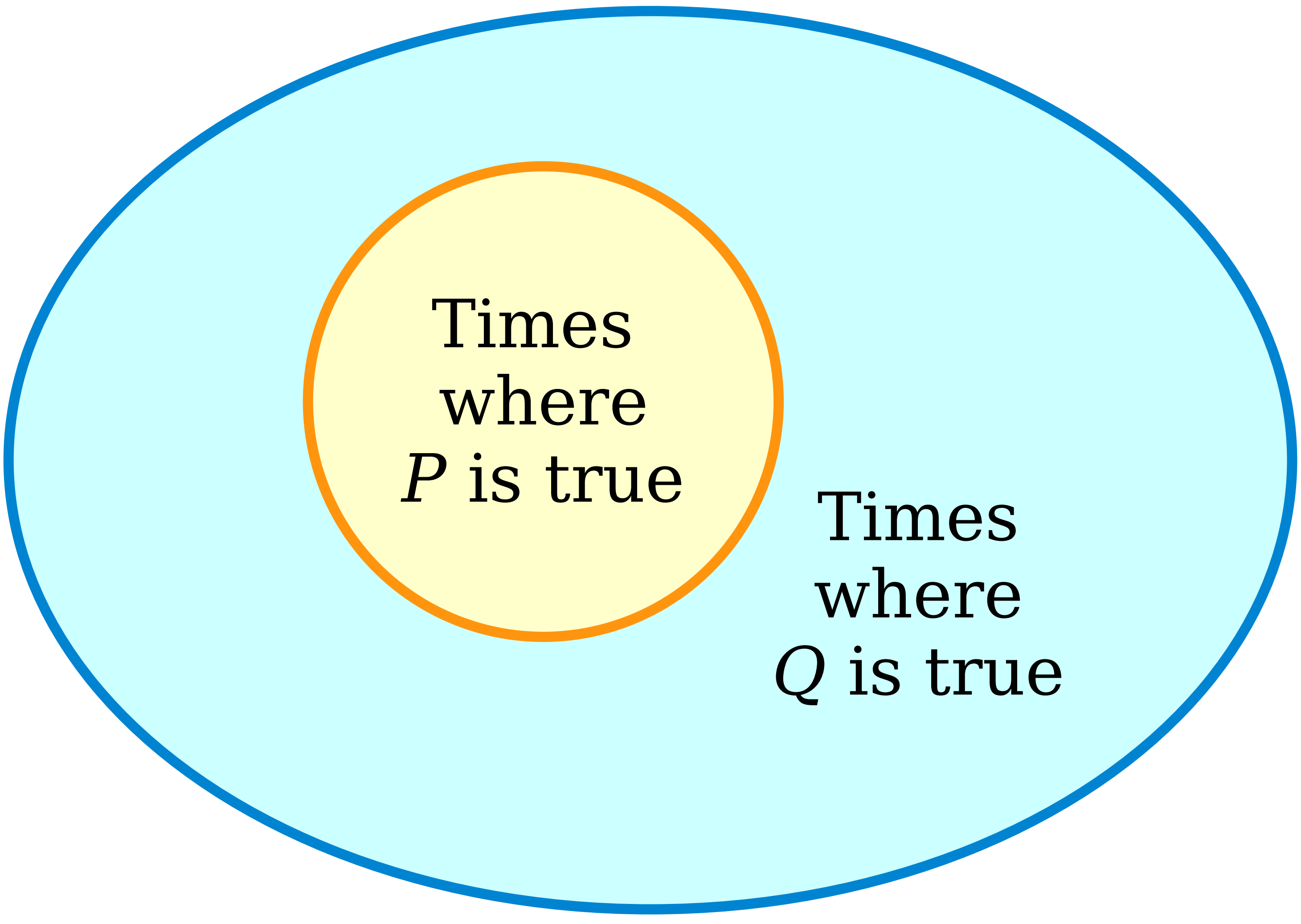
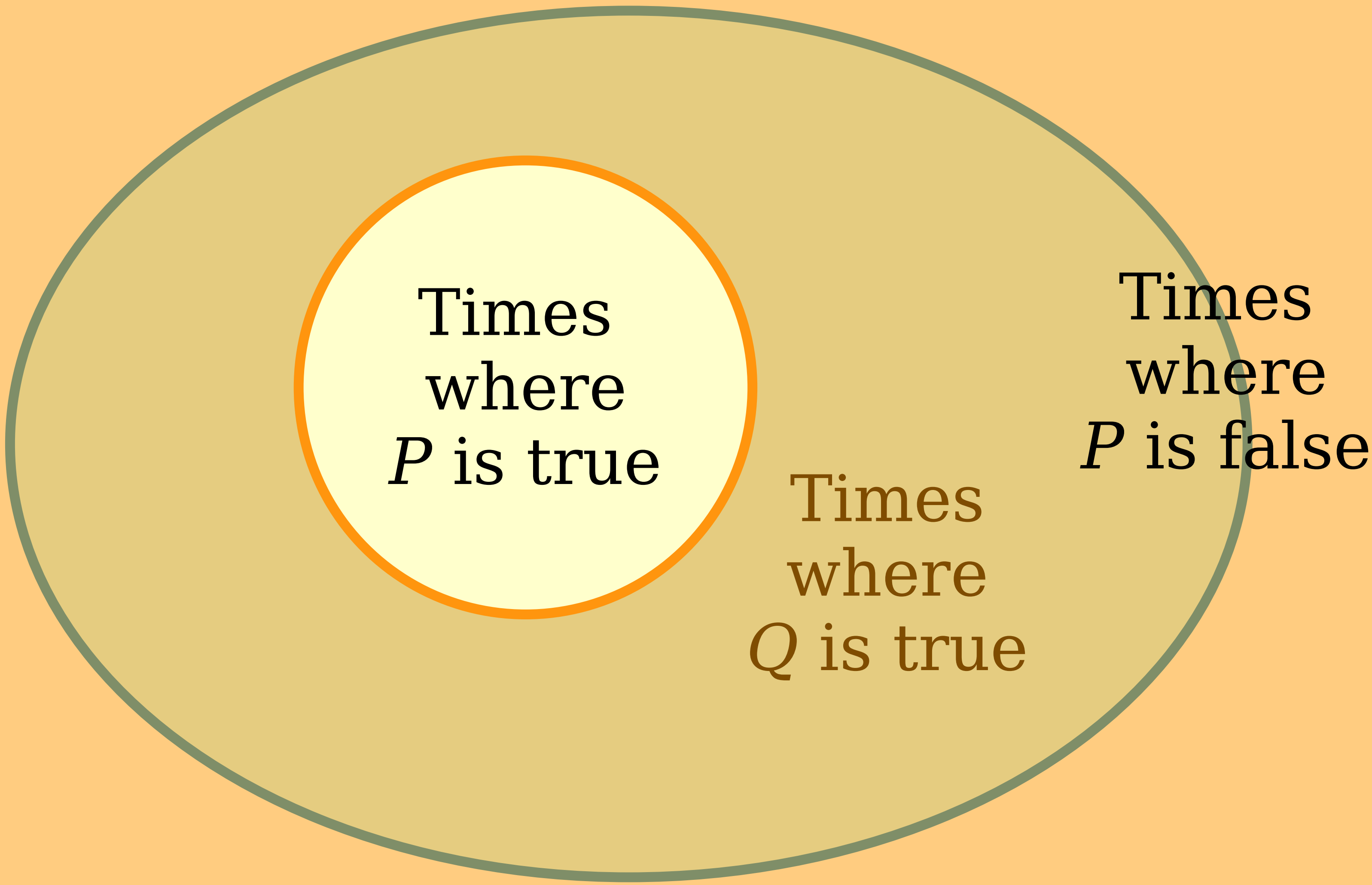


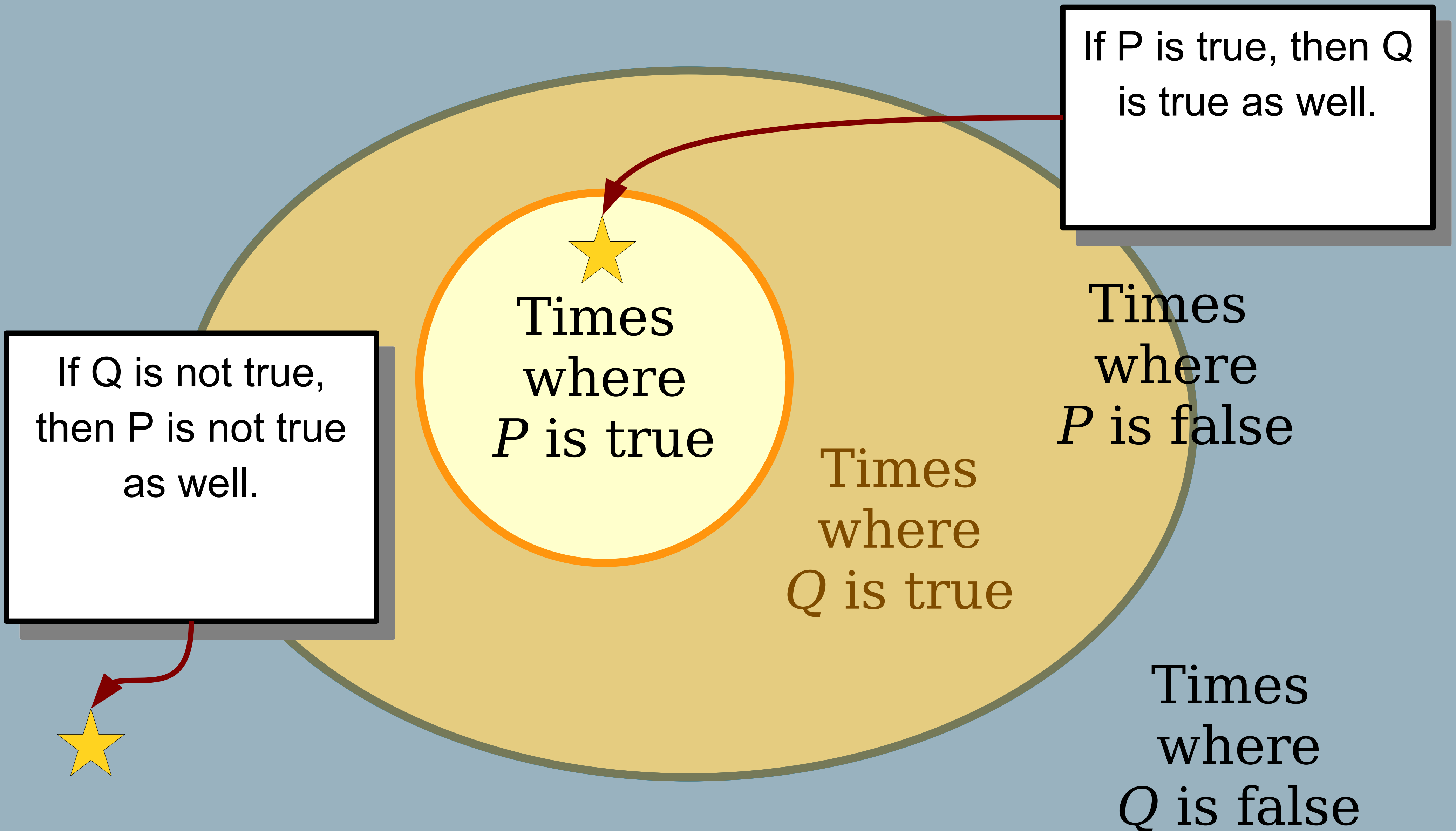
Implication, Diagrammatically



Implication, Diagrammatically



Implication, Diagrammatically



If P is true, then Q is true.

If Q is false, then P is false.

What are the negations of the above two statements?

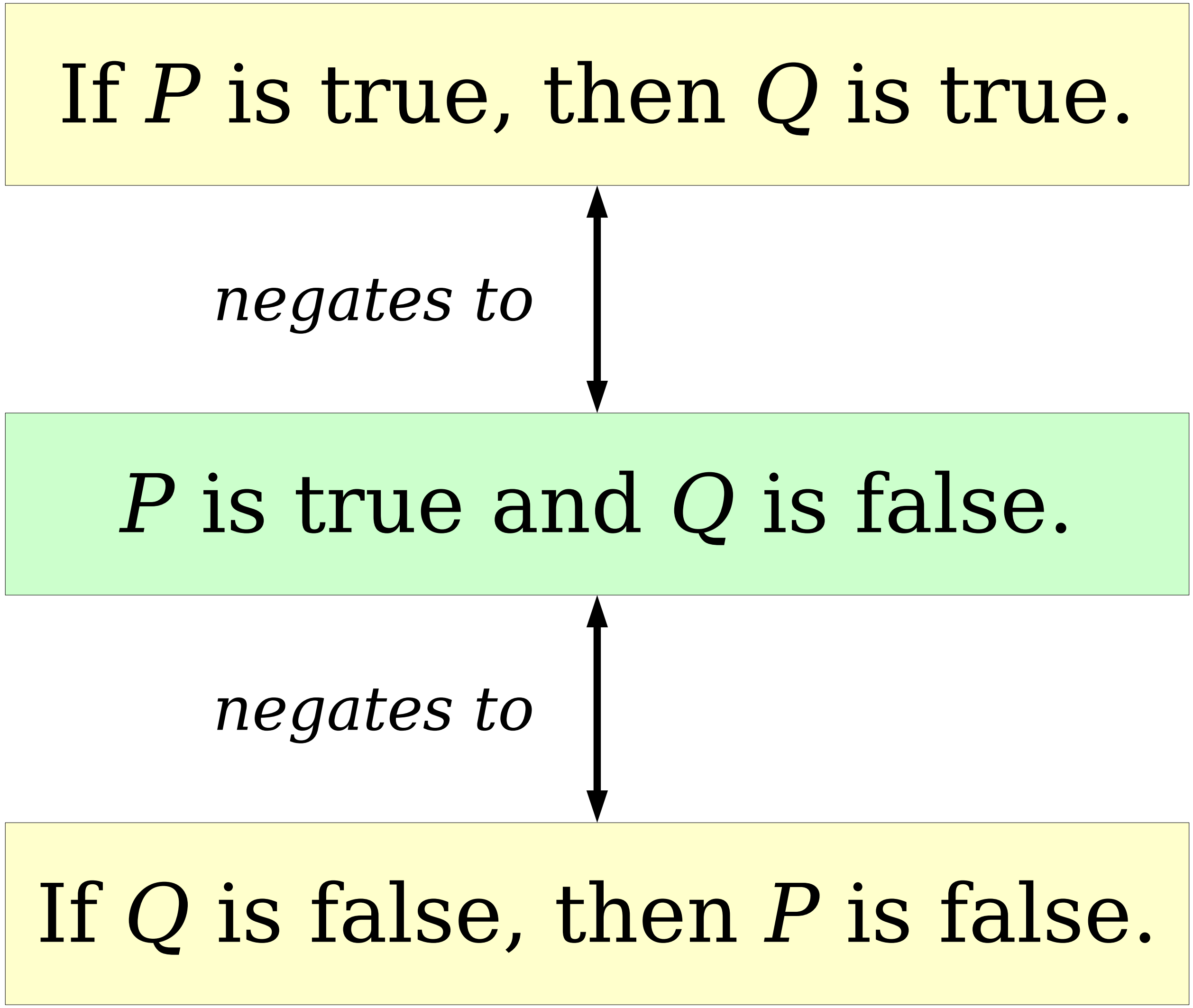
If P is true, then Q is true.

↕ *negates to*

P is true and Q is false.

If Q is false, then P is false.

What are the negations of the above two statements?



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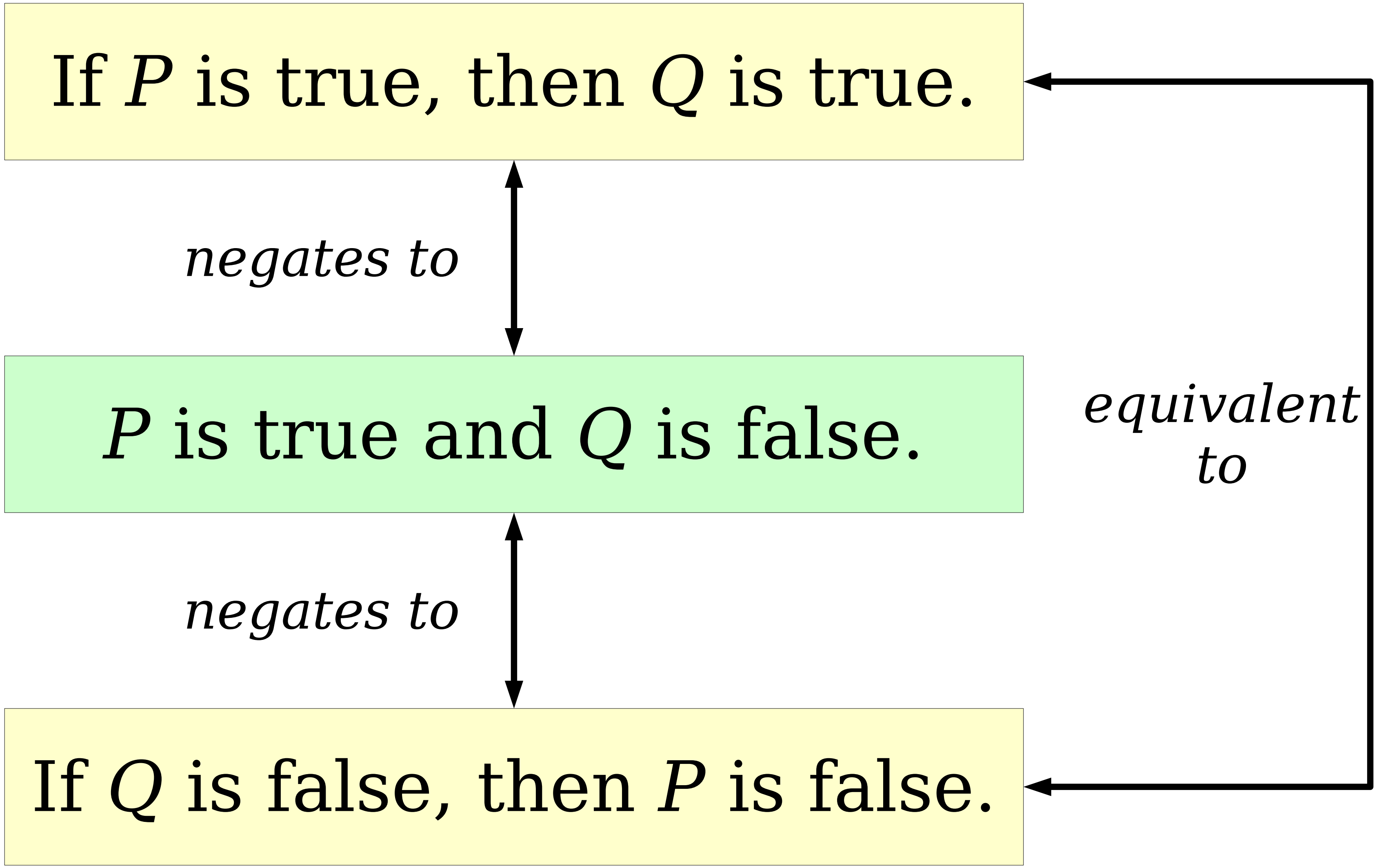


What are the negations of the above two statements?

The Contrapositive

- The ***contrapositive*** of the implication

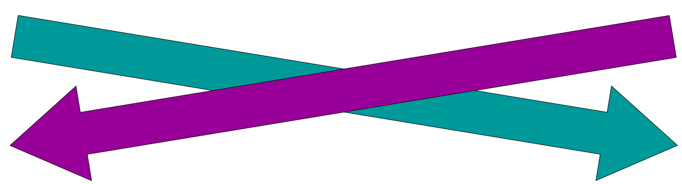
  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If it's a puppy, then I love it.



If I don't love it, then it's not a puppy.

The Contrapositive

- The ***contrapositive*** of the implication

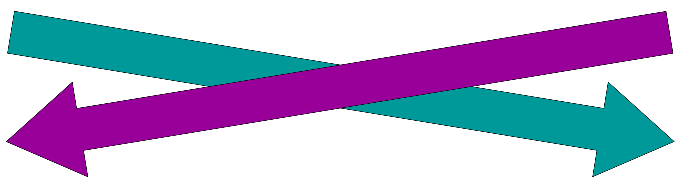
  If *P* is true, then *Q* is true

  is the implication

  If *Q* is false, then *P* is false.

- The contrapositive of an implication means exactly the same thing as the implication itself.

If I store cat food inside, then raccoons won't steal it.



If raccoons stole the cat food, then I didn't store it inside.

To prove the statement

"if *P* is true, then *Q* is true,"

you can choose to instead prove the equivalent statement

"if *Q* is false, then *P* is false,"

if that seems easier.

This is called a ***proof by contrapositive***.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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Proof: We will prove the contrapositive of this statement.

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Proof: We will prove the contrapositive of this statement.

This is a courtesy to the reader and says “heads up! we’re not going to do a regular old-fashioned direct proof here.”

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

Question: What is the contrapositive of this statement?

A) If n^2 is odd, then n is odd.

B) If n is odd, then n^2 is odd.

C) If n is even, then n^2 is even.

D) None of the above.

Respond at pollev.com/zhenglian740

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

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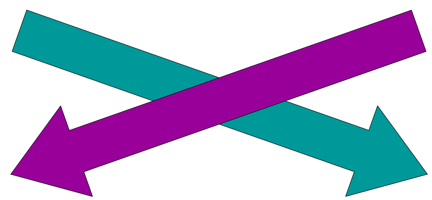

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

What is the contrapositive of this statement?

if n^2 is even, then n is even.

If n is odd, then n^2 is odd.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

Here, we're explicitly writing out the contrapositive. This tells the reader what we're going to prove. It also acts as a sanity check by forcing us to write out what we think the contrapositive is.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd.

We've said that we're going to prove this new implication, so let's go do it! The rest of this proof will look a lot like a standard direct proof.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

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Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$n^2 = (2k + 1)^2$$

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$$\begin{aligned} n^2 &= (2k + 1)^2 \\ &= 4k^2 + 4k + 1 \end{aligned}$$

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$$
\begin{aligned}
n^2 &= (2k + 1)^2 \\
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1.
\end{aligned}
$$

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

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From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

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We know integer us that

From th (namely means t to show. ■

The general pattern here is the following:

1. Start by announcing that we're going to use a proof by contrapositive so that the reader knows what to expect.

2. Explicitly state the contrapositive of what we want to prove.

3. Go prove the contrapositive.

Theorem: For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

Proof: We will prove the contrapositive of this statement, that if n is odd, then n^2 is odd. So let n be an arbitrary odd integer; we'll show that n^2 is odd as well.

We know that n is odd, which means there is an integer k such that $n = 2k + 1$. This in turn tells us that

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

From this, we see that there is an integer m (namely, $2k^2 + 2k$) such that $n^2 = 2m + 1$. That means that n^2 is odd, which is what we needed to show. ■

Biconditionals

- The previous theorem, combined with what we saw on Wednesday, tells us the following:

  For any integer *n*, if *n* is even, then n^2 is even.

  For any integer *n*, *i*f n^2 is even, then *n* is even.

- These are two different implications, each going the other way.
- We use the phrase ***if and only if*** to indicate that two statements imply one another.
- For example, we might combine the two above statements to say

  for any integer *n*: *n* is even if and only if n^2 is even.

Proving Biconditionals

- To prove a theorem of the form

  ***P* if and only if *Q*,**

  you need to prove two separate statements.
  - First, that if P is true, then Q is true.
  - Second, that if Q is true, then P is true.
- You can use any proof techniques you'd like to show each of these statements.
  - In our case, we used a direct proof for one and a proof by contrapositive for the other.

Let’s take a quick break!

Time-Out for Announcements!

Outdoor Activities Guide

- Being on campus means you're less than fifty miles from grassy mountains, redwood forests, Pacific coastline, beautiful wetlands, and more.
- Want to explore the area to see what it has to offer? Check out our (unofficial) Outdoor Activities Guide.

https://cs103.stanford.edu/outdoor_activities

- A sampler of what to check out:
  - Drive to the observatory in the mountains near San Jose and take in the views.
  - Visit a beach with an enormous colony of elephant seals.
  - Walk in redwood forests and pick your own bay leaves.
  - Stroll in the chaparral and see deer and wild turkeys.


California is breathtaking!!

Readings for Today

- On the course website we have some information you should look over.
- First is the ***Proofwriting Checklist***. It contains information about style expectations for proofs. We'll be using this when grading, so be sure to read it over.
- Next is the ***Guide to Office Hours***, which talks about how our office hours work and how to make the most effective use of them.
- Finally is the ***Guide to LaTeX***, which explains how to use LaTeX to typeset your problem sets in a way that's so beautiful it will bring tears to your eyes.

Problem Set One

- Problem Set Zero was due at 6:00PM today.
  - Missed the deadline? Ping us and we'll see what we can do.
- Problem Set One goes out today. It's due next Friday at 5:30PM.
  - Explore the language of set theory and better intuit how it works.
  - Learn more about the structure of mathematical proofs.
  - Write your first "freehand" proofs based on your experiences.
- As always, reach out if you have any questions!

Submitting Assignments

- All assignments should be submitted through GradeScope.
  - The programming portion of the assignment gets submitted separately from the written component.
  - The written component ***must*** be typed up; handwritten solutions don't scan well and get mangled in GradeScope.
- We don't do late days in CS103. Because submission times are recorded automatically, we're strict about the submission deadlines.
  - ***Very good idea:*** Leave at least two hours buffer time for your first assignment submission, just in case something goes wrong.
  - ***Very bad idea:*** Wait until the last minute to submit.
- However, we are pretty generous with how we grade. Your score on the problem sets is the square root of your raw score. So an 81% maps to a 90%, a 50% maps to a 71%, etc. This gives a huge boost even if you need to turn something in that isn't done.

Getting Help

- It is ***completely normal*** in this class to need to get help from time to time.
- Feel free to ask clarifying and conceptual questions on EdStem.
- Need more structured help? We have office hours! Feel free to stop on by.
  - Check out the online “Guide to Office Hours” for more information about how our office hours system works.
  - The OH calendar is available on the course website.

Working in Pairs

- You can work on problem sets individually or in pairs.
- Each person/pair should only submit a single problem set. In other words, if you're working in a pair, you and your partner should agree who will make the submission.
- For more details, check the Syllabus and Honor Code pages on the course website.

Finding a Problem Set Partner

Looking for a problem set partner?

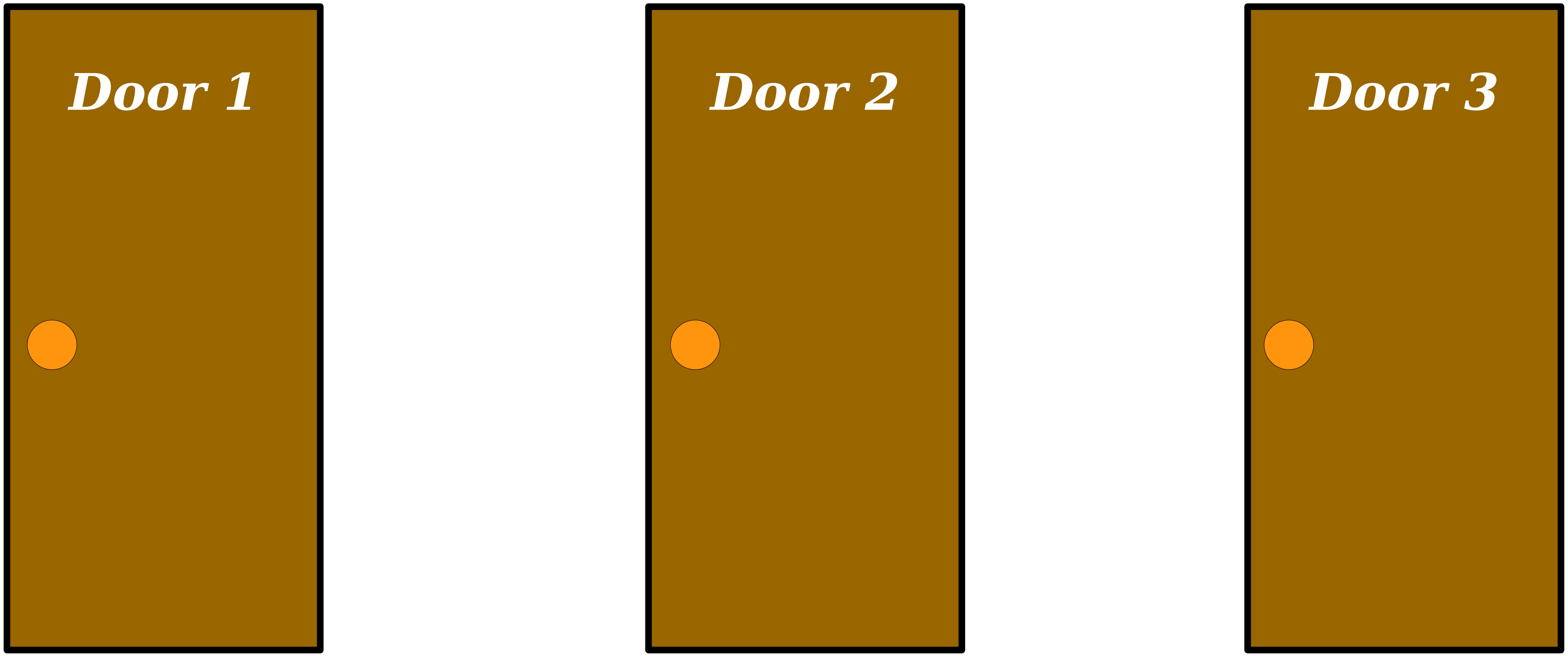
- Meet folks in lecture!
- Meet folks in office hours!
- Check out our pinned thread on EdStem!
- Fill out our matchmaking form!

A Note on the Honor Code

Back to CS103!

Proof by Contradiction

There's something hidden behind one of these doors. Which door is it hidden behind?



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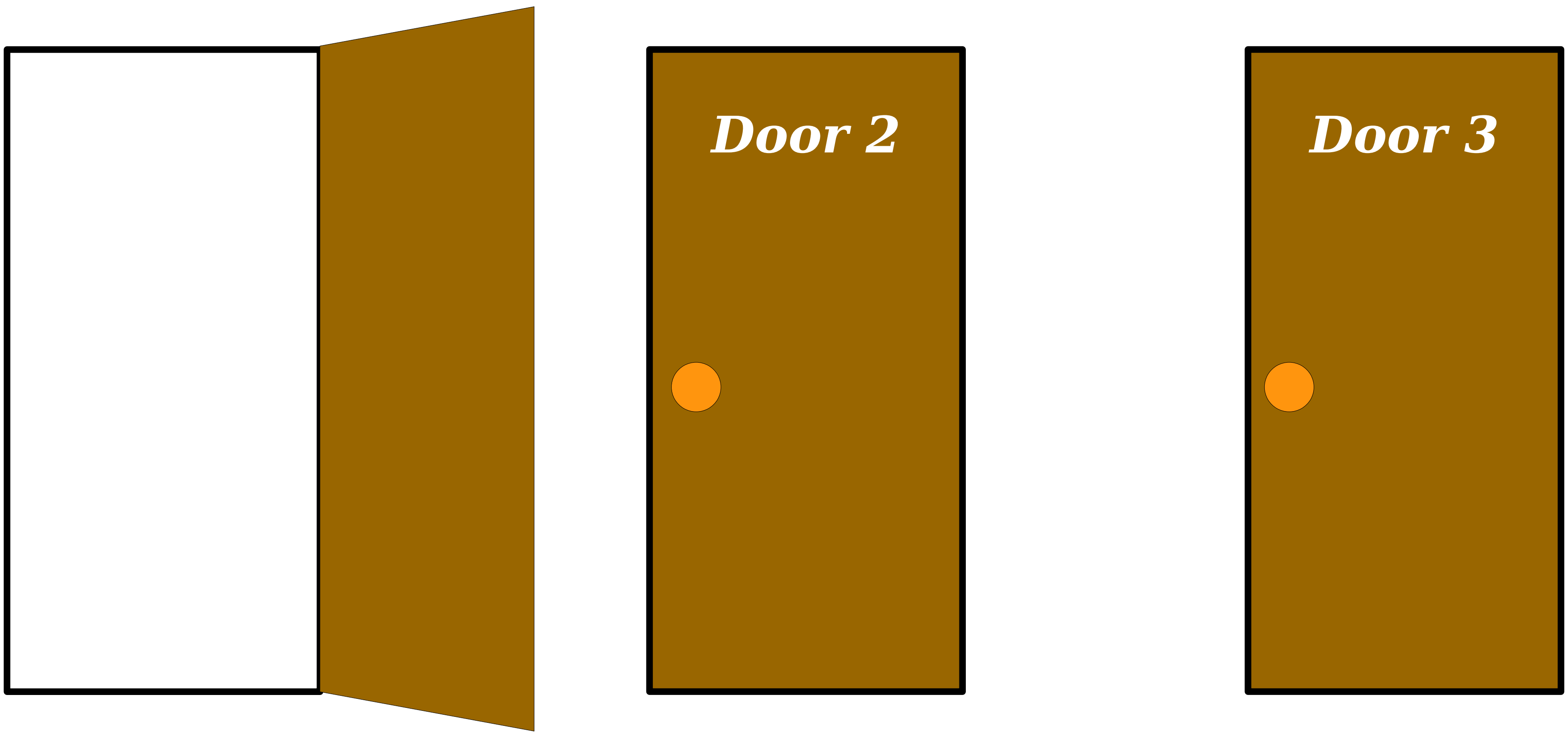

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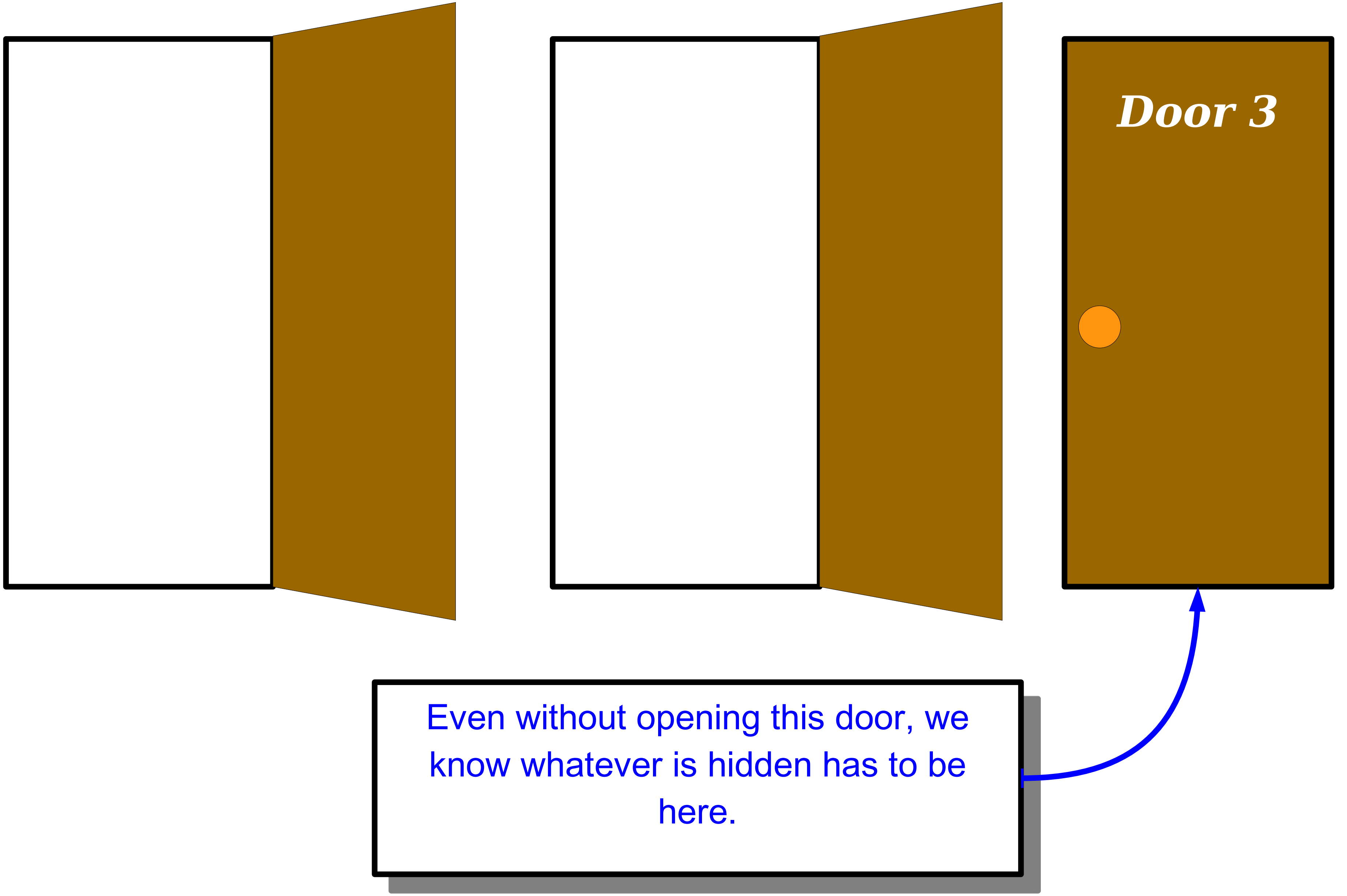

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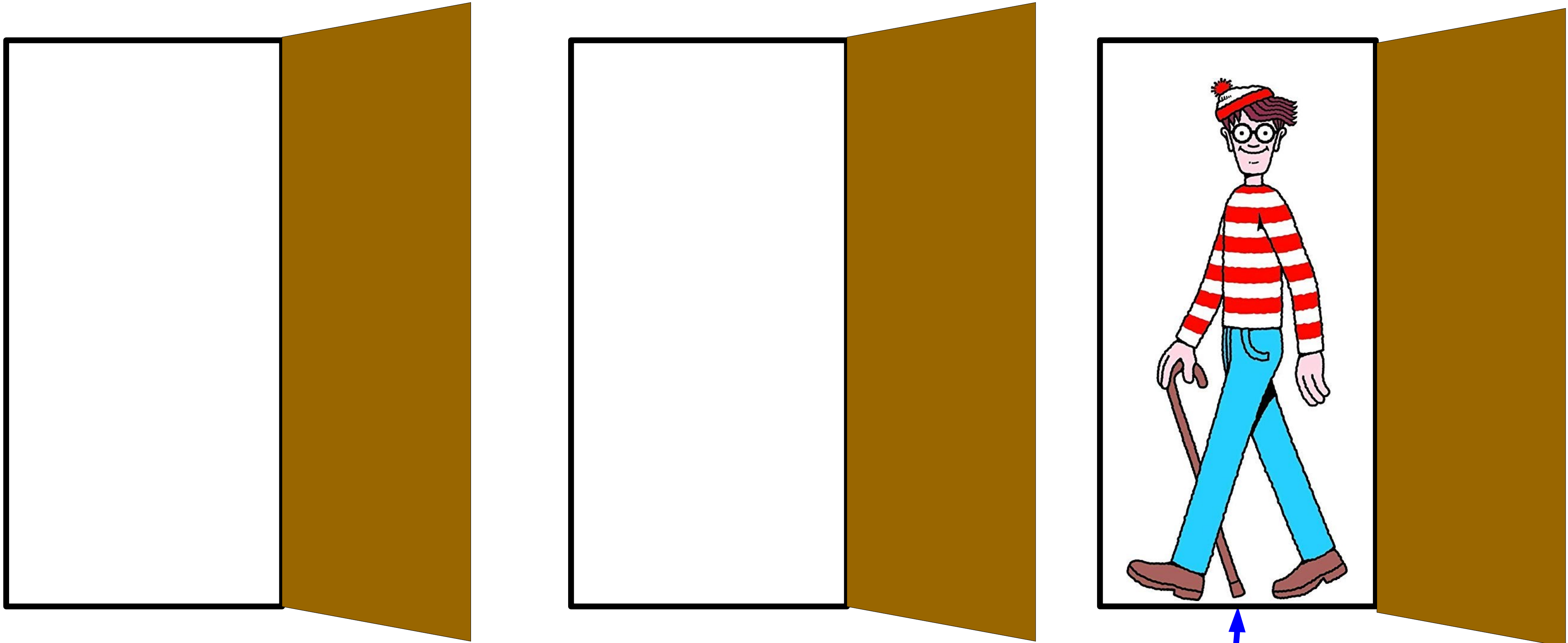
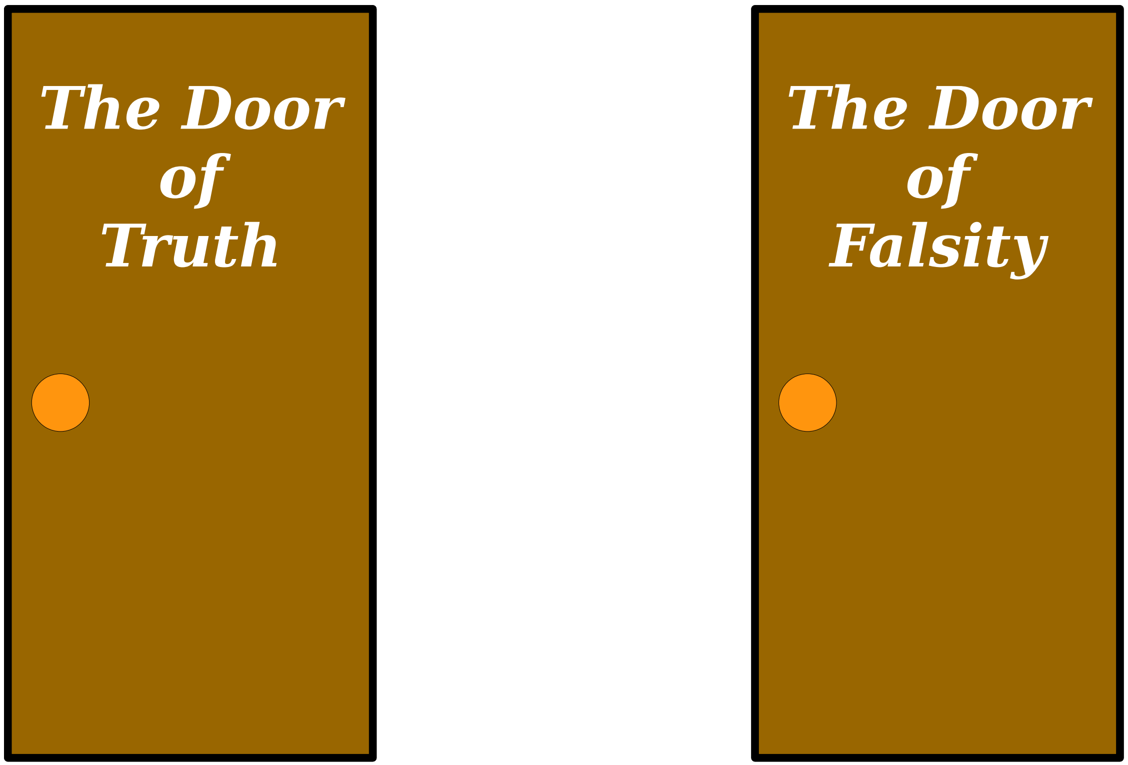


Even without opening this door, we know whatever is hidden has to be here.

Every statement in mathematics is either true or false.
If statement P is ***not false****, what does that tell you?*



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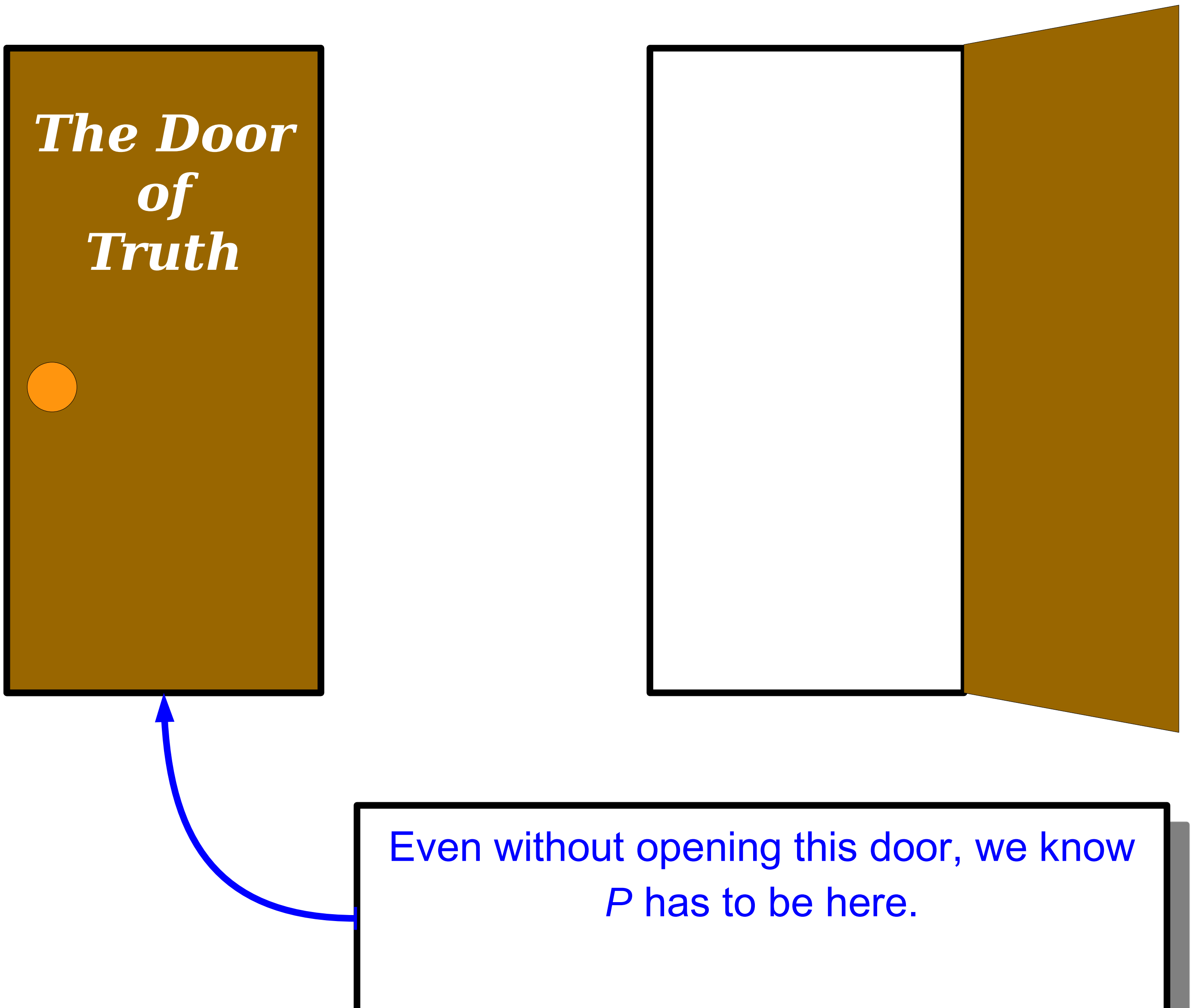

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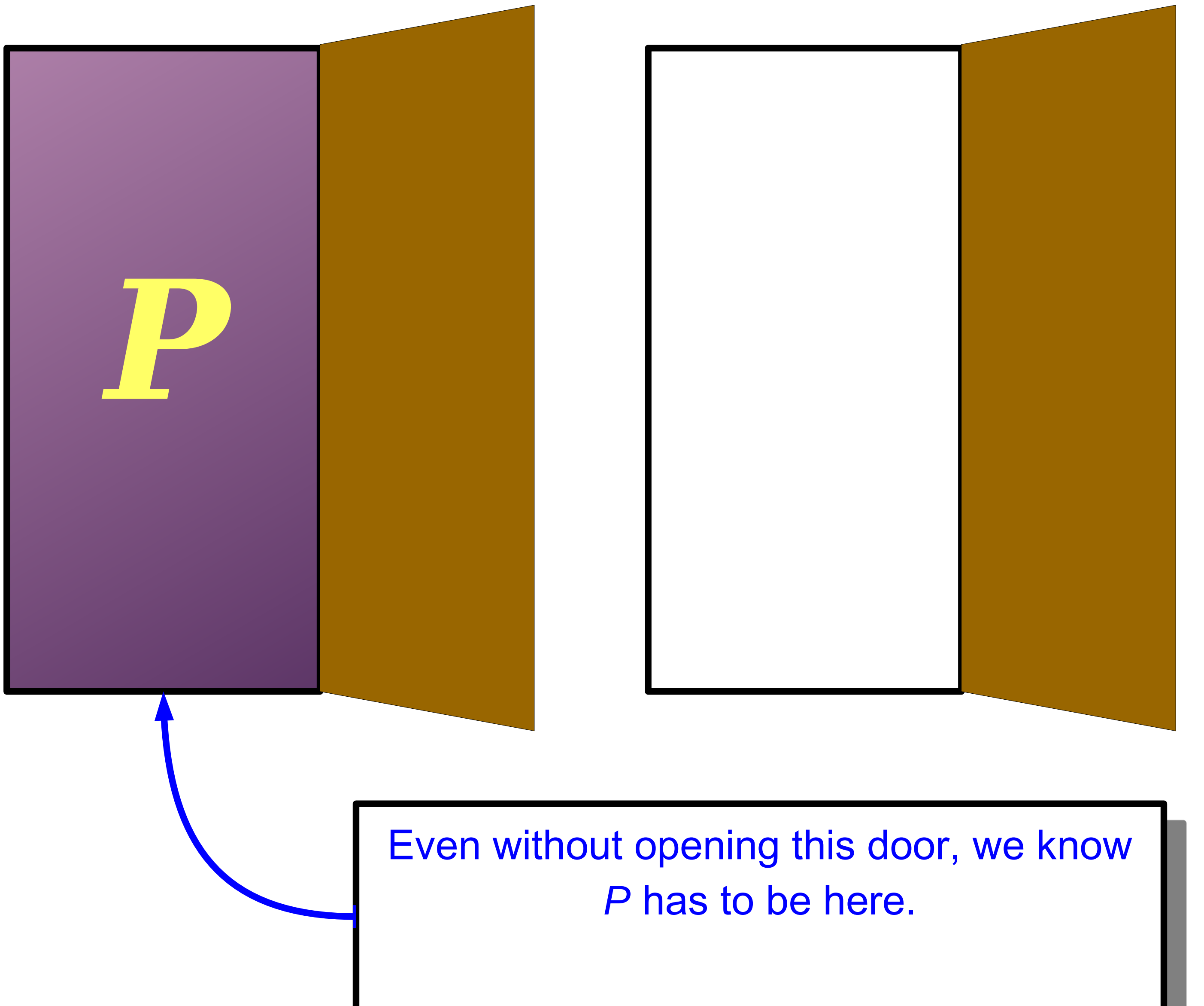

A ***proof by contradiction*** shows that some statement P is true by showing that P isn't false.

Proof by Contradiction

- ***Key Idea:*** Prove a statement P is true by showing that it isn't false.
- First, assume that P is false. The goal is to show that this assumption is silly.
- Next, show this leads to an impossible result.
  - For example, we might have that $1 = 0$, that $x \in S$ and $x \notin S$, that a number is both even and odd, etc.
- Finally, conclude that since P can't be false, we know that P must be true.

An Example: ***Set Cardinalities***

Set Cardinalities

- We've seen sets of many different cardinalities:
  - $|\varnothing| = 0$
  - $|\{1, 2, 3\}| = 3$
  - $|\{ n \in \mathbb{N} \mid n < 137 \}| = 137$
  - $|\mathbb{N}| = \aleph_0$.
- These span from the finite up through the infinite.
- ***Question:*** Is there a "largest" set? That is, is there a set that's bigger than every other set?

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One option: "there *is* a largest set."

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Notice that we're announcing

1. **that this is a proof by contradiction, and**
2. **what, specifically, we're assuming.**

This helps the reader understand where we're going. Remember – proofs are meant to be read by other people!

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The three key pieces:

1. Say that the proof is by contradiction.
2. Say what you are assuming is the negation of the statement to prove.
3. Say you have reached a contradiction and what the contradiction means.

In CS103, please include all these steps in your proofs!

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- Suppose we want to prove this implication:

  If *P* is true, then *Q* is true.

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    … what does this look like?

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Question: What is the negation of this statement?

Respond at pollev.com/zhenglian740

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Question: How do we complete this sentence?

Respond at pollev.com/zhenglian740

Theorem: For any integer n, if n^2 is even, then n is even.

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Since n is odd we know that there is an integer k such that

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Squaring both sides of equation (1) and simplifying gives the following:

$$n^2 = (2k + 1)^2$$

Question: What would the rest of this proof look like? Remember, we are trying to arrive at some sort of contradiction. What can we say about n^2?

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    Assume ***P is true*** and ***Q is false***, then derive a contradiction.

What We Learned

- ***What's an implication?***
  - It's statement of the form "if *P*, then *Q*," and states that if *P* is true, then *Q* is true.
- ***How do you negate formulas?***
  - It depends on the formula. There are nice rules for how to negate universal and existential statements and implications.
- ***What is a proof by contrapositive?***
  - It's a proof of an implication that instead proves its contrapositive.
  - (The contrapositive of "if *P*, then *Q*" is "if not *Q*, then not *P*.")
- ***What's a proof by contradiction?***
  - It's a proof of a statement *P* that works by showing that *P* cannot be false.

Your Action Items

- ***Read "Guide to Office Hours," the "Proofwriting Checklist," and the "Guide to LaTeX."***
  - There's a lot of useful information there. In particular, be sure to read the Proofwriting Checklist, as we'll be working through this checklist when grading your proofs!
- ***Start working on PS1.***
  - At a bare minimum, read over it to see what's being asked. That'll give you time to turn things over in your mind this weekend.

Next Time

- ***Mathematical Logic***
  - How do we formalize the reasoning from our proofs?
- ***Propositional Logic***
  - Reasoning about simple statements.
- ***Propositional Equivalences***
  - Simplifying complex statements.